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Comments on superstring interactions in a plane-wave background

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Abstract: The three string vertex for type-IIB superstrings in a maximally supersymmetric plane-wave background is investigated. Specifically, we derive a factorization theorem for the Neumann coefficients that generalizes a flat-space result that was obtained some 20 years ago. The resulting formula is used to explore the leading large μ asymptotic behavior, which is relevant for comparison with dual gauge theory results.

Keywords: Superstrings and Heterotic Strings, String Field Theory, Penrose limit and pp-wave background.

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Contents

1. Introduction

It was recently discovered that type-IIB superstring theory admits a maximally supersymmetric plane-wave background [1]. Moreover, the string theory in this background is tractable, despite the fact that the background contains a nonvanishing RR field, in the light-cone GS formalism [2]. In that approach the world-sheet theory consists of free massive bosons and fermions. Subsequently, there was a proposal to relate the string states and their interactions to holographically dual calculations in terms of certain limiting operators and their correlation functions in $\mathcal{N} = 4$ super Yang-Mills theory [3].

The string interactions are encoded (using the language of light-cone-gauge string field theory) in a cubic interaction vertex. This vertex has been formulated in a pair of very nice papers by Spradlin and Volovich [4, 5]. Their work generalizes the flat-space light-cone-gauge field theory results of [6, 7] to the plane-wave geometry.

This paper has two goals. First, we wish to make the formulas for the Neumann coefficients that enter in the interaction vertex more explicit. Specifically, one wants explicit formulas for the inverse of a certain infinite-dimensional matrix. We succeed in expressing the inverse matrix in a factorized form in terms of a certain infinite component vector, but we have not yet obtained explicit formulas for the vector. However, even this step is useful for our second goal: exploring the large μ (large curvature) limit of the geometry, which is required for making contact with perturbative gauge theory computations.

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2. Review of basic formulas

The type-IIB superstring in the maximally supersymmetric plane-wave background is described in light-cone gauge by a free world-sheet theory. The eight bosonic and eight fermionic world-sheet fields each have mass μ, a parameter that enters in the description of the plane-wave geometry and the RR five-form field strength. The mass term has two important consequences. One is that it leads to a mixing of left-movers and right-movers. The other is that the zero modes are also described by harmonic oscillators of finite frequency. Altogether, a convenient labeling of the bosonic lowering and raising operators arising from quantization of the free world-sheet theory is a_m^I and $a_m^{I\dagger}$, where m runs from minus infinity to plus infinity and $I = 1, \ldots, 8$. These satisfy ordinary oscillator commutation relations

$$[a_m^I, a_n^{J\dagger}] = \delta_{mn}\delta^{IJ} \,. \tag{2.1}$$

There are also fermionic oscillators b_m^α and $b_m^{\alpha\dagger}$, which will not be discussed in this paper.

The spectrum of the free string theory is described by the light-cone Hamiltonian

$$H_2 = \sum_{m=-\infty}^{\infty} \omega_m N_m \tag{2.2}$$

where N_m is the number of excitations of level m oscillators

$$N_m = \sum_{I=1}^{8} a_m^{I\dagger} a_m^I + \text{fermionic terms} \tag{2.3}$$

where the frequencies are given by

$$\omega_m = \sqrt{m^2 + \mu^2\alpha^2} \,. \tag{2.4}$$

The second term in the square root is actually $(\alpha'\mu p_-)^2$, but we set the Regge slope $\alpha' = 2$ and define $\alpha = 2p_-$. (In flat space, we used the symbol p^+ rather than p_-, but in curved space a lower index is preferable.) The physical spectrum is given by the product of all the oscillator spaces subject to one constraint

$$\sum_{m=-\infty}^{\infty} m N_m = 0 \,. \tag{2.5}$$

In flat space this constraint reduces to the usual level-matching condition for left-movers and right-movers.

The three-string interaction vertex for type-IIB superstrings in flat space was worked out in [6] and [7] and generalized to the plane-wave geometry in [4] and [5]. The formula can be written rather elegantly in terms of functionals, but to make its meaning precise it is desirable to expand it out in terms of oscillators. A convenient notation is to use a tensor product of three string Fock spaces, labeled by an index $r = 1, 2, 3$. Then the three string interaction vertex contains a factor

$$|V_B\rangle = \exp\left(\frac{1}{2}\sum_{r,s=1}^{3}\sum_{m,n=-\infty}^{\infty}\sum_{I=1}^{8} a_{mr}^{I\dagger}\bar{N}_{mn}^{rs} a_{ns}^{I\dagger}\right)|0\rangle \,. \tag{2.6}$$

The quantities $\bar{N}^{rs}_{mn}$, called Neumann coefficients, are the main objects of concern in this paper. Their definition here differs from that used in [6] and [7] by factors of $\sqrt{mn}$. The definition given here is more natural for the $\mu \neq 0$ generalization. The three string vertex also contains a similar expression $|\,V_F\rangle$ made out of the fermionic oscillators and a "prefactor" that is polynomial in the various oscillators. We will not consider either of these in this paper.

In describing the Neumann matrices, it is convenient to consider separately the cases in which each of the indices m, n are either positive, negative or zero. Henceforth, the symbols m, n will always denote positive integers. Negative integers will be indicated by displaying an explicit minus sign. One result of [4], for example, using matrix notation for the blocks with positive indices, is

$$\bar{N}^{rs} = 1 - 2(C_r C^{-1})^{1/2} A^{(r)T} \Gamma_+^{-1} A^{(s)} (C_s C^{-1})^{1/2}\,. \tag{2.7}$$

Here $C_{mn} = m\delta_{mn}$ and $(C_r)_{mn} = \omega_{rm}\delta_{mn}$, where $\omega_{rm} = \sqrt{m^2 + (\mu\alpha_r)^2}$. These are simple diagonal matrices. The great challenge is to understand the rest of the formula. The definitions of $A^{(r)}$ and Γ_+, and other expressions that appear here, are collected in the appendix.

The blocks with both indices negative are related in a simple way to the ones with both indices positive by

$$\bar{N}^{rs}_{-m-n} = -\left(U_r \bar{N}^{rs} U_s\right)_{mn}, \tag{2.8}$$

where

$$U_r = C^{-1}(C_r - \mu\alpha_r)\,. \tag{2.9}$$

In the case of $\bar{N}^{33}$ these are the only nonvanishing terms. For the other Neumann coefficients the other nonvanishing terms are

$$\bar{N}^{3r}_{m0} = \bar{N}^{r3}_{0m} = \sqrt{2\mu\alpha_r}\,\epsilon^{rs}\alpha_s \left[\left(C_3 C^{-1}\right)^{1/2} \Gamma_+^{-1} B\right]_m \qquad r, s = 1, 2 \tag{2.10}$$

$$\bar{N}^{rs}_{m0} = bar N^{sr}_{0m} = \sqrt{2\mu\alpha_s}\,\epsilon^{st}\alpha_t \left[\left(C_r C^{-1}\right)^{1/2} A^{(r)T}\Gamma_+^{-1} B\right]_m \qquad r, s, t = 1, 2 \tag{2.11}$$

and

$$\bar{N}^{rs}_{00} = \delta^{rs} + \frac{\sqrt{\alpha_r\alpha_s}}{\alpha_3} - \mu\sqrt{\alpha_r\alpha_s}\,\epsilon^{rt}\epsilon^{su}\alpha_t\alpha_u B^T \Gamma_+^{-1} B \qquad r, s, t, u = 1, 2 \tag{2.12}$$

$$\bar{N}^{3r}_{00} = \bar{N}^{r3}_{00} = -\sqrt{-\frac{\alpha_r}{\alpha_3}} \qquad r = 1, 2\,. \tag{2.13}$$

The asymmetry between string number three and the other two strings is a reflection of the fact that the μ dependence of the formula breaks the cyclic symmetry that is present in the flat space case. Evidently, crossing symmetry no longer holds in the plane-wave geometry.

To make the formulas useful for comparison with the dual gauge theory, it would be helpful to have explicit formulas in which the various matrix multiplications and inversions have been analytically evaluated. The quantities that we especially would like to evaluate explicitly are the matrix Γ_+^{-1}, the vector

$$Y = \Gamma_+^{-1} B\,, \tag{2.14}$$

and the scalar

$$k = B^T \Gamma_+^{-1} B \,. \tag{2.15}$$

In the case of flat space ($\mu = 0$) the results are known. Specifically

$$\bar{N}^{rs}_{mn} = -\frac{mn\alpha}{m\alpha_s + n\alpha_r} \bar{N}^r_m \bar{N}^s_n \qquad \text{for} \quad \mu = 0 \tag{2.16}$$

where $\alpha = \alpha_1\alpha_2\alpha_3$ and

$$\bar{N}^r_m = \frac{\sqrt{m}}{\alpha_r} f_m \left(-\frac{\alpha_{r+1}}{\alpha_r} \right) e^{m\tau_0/\alpha_r} \qquad \text{for} \quad \mu = 0 \,, \tag{2.17}$$

$$f_m(\gamma) = \frac{\Gamma(m\gamma)}{m!\Gamma(m\gamma + 1 - m)} \tag{2.18}$$

and

$$\tau_0 = \sum_{r=1}^{3} \alpha_r \log|\alpha_r| \,. \tag{2.19}$$

In particular, still for $\mu = 0$, $\Gamma_+^{-1} = \frac{1}{2}(1 - \bar{N}^{33})$, $Y_m = -\bar{N}^3_m$, and $k = 2\tau_0/\alpha$.

3. Neumann coefficients for $\mu \neq 0$

In this section we will derive the generalization of eq. (2.16) that holds for the plane-wave geometry. The method of derivation is a fairly straightforward generalization of the one used for flat space in [6]. We begin by defining

$$\tilde{\Gamma}_+ = \sum_{r=1}^{3} A^{(r)} U_r^{-1} A^{(r)T} \tag{3.1}$$

and then considering the product

$$\Gamma_+ C^{-1} \tilde{\Gamma}_+ = \left(U_3 + \sum_1^2 A^{(r)} U_r A^{(r)T} \right) C^{-1} \left(U_3^{-1} + \sum_1^2 A^{(s)} U_s^{-1} A^{(s)T} \right) . \tag{3.2}$$

Using various identities given in the appendix, this simplifies to

$$\Gamma_+ C^{-1} \tilde{\Gamma}_+ = U_3 C^{-1} \tilde{\Gamma}_+ + \Gamma_+ C^{-1} U_3^{-1} - \frac{1}{2} \alpha_1 \alpha_2 B B^T \,. \tag{3.3}$$

The next step is to use eq. (A.7) and (A.18) to deduce that

$$\tilde{\Gamma}_+ = \Gamma_+ + \mu\alpha B B^T \,. \tag{3.4}$$

Substituting this into the previous equation and multiplying left and right by Γ_+^{-1} gives

$$C^{-1} U_3^{-1} \Gamma_+^{-1} + \Gamma_+^{-1} U_3 C^{-1} = C^{-1} + \frac{1}{2} \alpha_1 \alpha_2 Y Y^T + \mu\alpha Z Y^T \tag{3.5}$$

where we have defined

$$Z = (1 - \Gamma_+^{-1} U_3) C^{-1} B \,. \tag{3.6}$$

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The next step is to eliminate Z from eq. (3.5). This is achieved by multiplying the equation on the right with the vector B. This gives a linear equation for Z, whose solution is

$$Z = \frac{1}{1+\mu\alpha k}\left(C^{-1}U_3^{-1} - \frac{1}{2}\alpha_1\alpha_2 k\right)Y\,. \tag{3.7}$$

Substituting this back into eq. (3.5) and simplifying gives the formula

$$\{\Gamma_+^{-1}, C_3\} = C + \frac{1}{2}\frac{\alpha_1\alpha_2}{1+\mu\alpha k}CU_3^{-1}YY^TCU_3^{-1}\,. \tag{3.8}$$

If we had explicit formulas for the vector Y and the scalar k, this formula would give us the matrix Γ_+^{-1}. It can be recast as a formula for the Neumann coefficient matrix $\bar{N}^{33}_{mn}$. The result is

$$\bar{N}^{33}_{mn} = -\frac{mn\alpha_1\alpha_2}{1+\mu\alpha k}\frac{\bar{N}^3_m\bar{N}^3_n}{\omega_{3m}+\omega_{3n}} \tag{3.9}$$

where

$$\bar{N}^3_m = -\left[(C^{-1}C_3)^{1/2}U_3^{-1}Y\right]_m\,. \tag{3.10}$$

Some further simple manipulations give the generalization

$$\bar{N}^{rs}_{mn} = -\frac{mn\alpha}{1+\mu\alpha k}\frac{\bar{N}^r_m\bar{N}^s_n}{\alpha_s\omega_{rm}+\alpha_r\omega_{sn}} \tag{3.11}$$

where

$$\bar{N}^r_m = -\left[(C^{-1}C_r)^{1/2}U_r^{-1}A^{(r)T}Y\right]_m\,. \tag{3.12}$$

This is the desired generalization of the flat-space formula eq. (2.16). However, we are still lacking a generalization of the explicit formula (2.17) as well as an explicit formula for k.

4. An involution

We are primarily interested in the case in which the mass parameter μ is positive. In particular, in the next section we will explore asymptotic properties for μ large and positive. The formulas for μ large and negative are different. We can be quite explicit about this by relating the two cases. For this purpose we define $\tilde{\Gamma}_+ = \Gamma_+(-\mu)$, $\tilde{Y} = Y(-\mu)$, and $\tilde{k} = k(-\mu)$. The expression $\tilde{\Gamma}_+$ was already introduced in eq. (3.1). In fact, we found that

$$\tilde{\Gamma}_+ = \Gamma_+ + \mu\alpha BB^T\,. \tag{4.1}$$

This equation can be inverted to give

$$\tilde{\Gamma}_+^{-1} = \Gamma_+^{-1} - \frac{\mu\alpha}{1+\mu\alpha k}YY^T\,. \tag{4.2}$$

Multiplying by B on the right one deduces that

$$\tilde{Y} = \frac{1}{1+\mu\alpha k}Y \tag{4.3}$$

and

$$\tilde{k} = \frac{k}{1 + \mu\alpha k}\,. \tag{4.4}$$

These formulas are of interest because they allow for nontrivial checks of various formulas. Each one must transform into another correct equation under the transformation $\mu \to -\mu$. We have checked this in every case, and no new equations beyond those already presented are generated in this way.

5. Large μ asymptotics

In the duality between string theory and gauge theory, it is necessary to consider large μ in order to make contact with perturbative gauge theory calculations. Therefore in this section we shall work out some of the leading terms in the asymptotic expansions of Γ_+^{-1}, $Y = \Gamma_+^{-1}B$, and $k = B^T Y = B^T \Gamma_+^{-1} B$. Some preliminary studies of these expansions, which we will extend, have been made previously in [8, 5, 9].

As will become evident, the leading (large μ) term in the expansion of Γ_+^{-1} is given by the first term on the right-hand side of eq. (3.8). Therefore let us extract this term by defining

$$\Gamma_+^{-1} = \frac{1}{2} C C_3^{-1} + R\,. \tag{5.1}$$

The first term has the expansion

$$\frac{1}{2} C C_3^{-1} \to -\frac{C}{2\mu\alpha_3} + \frac{C^3}{4(\mu\alpha_3)^3} + \cdots\,. \tag{5.2}$$

We will find that the leading term in R is of order μ^{-4}.

To analyze the asymptotic behavior for large positive μ, let us begin by inserting eq. (5.1) into eq. (3.8). This gives

$$\{R, C_3\} = \frac{1}{2} \frac{\alpha_1\alpha_2}{1 + \mu\alpha k} C U_3^{-1} Y Y^T C U_3^{-1}\,. \tag{5.3}$$

We now making an ansatz for the large μ structure of R, with an unknown coefficient. Specifically, let us try

$$R \to a_R \frac{\pi}{(\mu\alpha_3)^4} \left(\frac{\alpha_1\alpha_2}{\alpha_3}\right)^2 C^3 B B^T C^3 + \cdots\,. \tag{5.4}$$

This term is of order μ^{-4}. The next term in the expansion would be of order μ^{-6}. In similar fashion one can argue that at large μ

$$k = B^T Y \to -\frac{1}{\mu\alpha} - \frac{a_k}{\pi(\mu\alpha_1\alpha_2)^2} + \cdots\,. \tag{5.5}$$

Inserting these expansions into eq. (5.3), one learns that

$$a_R a_k = \frac{1}{64}\,. \tag{5.6}$$

While this relation is easy to derive, it is much more difficult to determine a_R and a_k separately.[1]

Let us consider now the asymptotic expansion of Y. In view of the above equations, it should have the structure

$$Y = \Gamma_+^{-1} B \to \frac{1}{\mu\alpha_3}\left[-\frac{1}{2}C + \left(\frac{1}{4} - x\right)\frac{C^3}{(\mu\alpha_3)^2} + \cdots\right] B\,. \tag{5.7}$$

Note that the leading term is of order μ^{-1} and the first correction is of order μ^{-3}.

The value of x is of particular interest, since a certain dual field theory calculation gives a result that is proportional to $1/2 - 4x$. Based on the light-cone field theory formulas described here, it was estimated numerically to be approximately $1/16$ in [9], and that is presumably correct. The field theory analysis of [10] gave a nonvanishing result corresponding to the value $x = 0$. Daniel Freedman informs me that he has repeated the field theory calculation, using a different regularization method, and that he finds a vanishing result, corresponding to the value $x = 1/8$. I hope to obtain an exact analytic expression for Y from which one could read off the exact value of x, but as yet this has not been achieved.

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A. Definitions and identities

The light cone momenta in the three-string vertex are proportional to α_r, where we take $\alpha_1, \alpha_2 > 0$, $\alpha_3 < 0$ and $\alpha_1 + \alpha_2 + \alpha_3 = 0$. We also define $\beta = \alpha_1/\alpha_3$, which satisfies $-1 < \beta < 0$.

The matrices $A^{(r)}_{mn}$, which appear in the Neumann coefficients, are given by

$$A^{(1)}_{mn} = \frac{2}{\pi}(-1)^{m+n+1}\sqrt{mn}\frac{\beta \sin m\pi\beta}{n^2 - m^2\beta^2}\,, \tag{A.1}$$

$$A^{(2)}_{mn} = \frac{2}{\pi}(-1)^{m+1}\sqrt{mn}\frac{(\beta+1)\sin m\pi\beta}{n^2 - m^2(\beta+1)^2}\,, \tag{A.2}$$

[1] In the first version of this paper I claimed to prove that $a_R = a_k = x = 1/8$. However, this is not correct. I am grateful to the authors of [9] for bringing the error to my attention.

and $A^{(3)}_{mn} = \delta_{mn}$. The indices m, n range from 1 to infinity. Additional quantities that we need are

$$B_m = \frac{2\alpha_3}{\pi\alpha_1\alpha_2}(-1)^{m+1}\frac{\sin m\pi\beta}{m^{3/2}} \tag{A.3}$$

and

$$C_{mn} = m\delta_{mn} \,. \tag{A.4}$$

Additional matrices that involve the mass parameter μ of the plane-wave geometry are

$$(C_r)_{mn} = \omega_{rm}\delta_{mn} = \sqrt{m^2 + \mu^2\alpha_r^2}\,\delta_{mn} \tag{A.5}$$

and

$$U_r = C^{-1}(C_r - \mu\alpha_r). \tag{A.6}$$

Note that

$$U_r^{-1} = C^{-1}(C_r + \mu\alpha_r) = U_r + 2\mu\alpha_r C^{-1}. \tag{A.7}$$

A crucial construct is the infinite matrix

$$\Gamma_+ = \sum_{r=1}^{3} A^{(r)} U_r A^{(r)T}. \tag{A.8}$$

Explicit formulas for its inverse are a main goal of our work. Related quantities that also are needed are the infinite vector

$$Y = \Gamma_+^{-1} B \tag{A.9}$$

and the scalar

$$k = B^T \Gamma_+^{-1} B \,. \tag{A.10}$$

The infinite matrices $A^{(r)}_{mn}$ and the infinite vector B_m satisfy a number of useful relations which we record here

$$A^{(r)T} C A^{(s)} = -\frac{\alpha_3}{\alpha_r} C \delta^{rs} \qquad r, s = 1, 2 \tag{A.11}$$

$$A^{(r)T} C^{-1} A^{(s)} = -\frac{\alpha_r}{\alpha_3} C^{-1} \delta^{rs} \qquad r, s = 1, 2 \,. \tag{A.12}$$

The symbol T means matrix transpose.

$$B^T C B = \frac{2}{\alpha_1\alpha_2} \tag{A.13}$$

$$B^T C^{-1} B = \frac{2\pi^2}{3\alpha_3^2} \tag{A.14}$$

$$B^T \frac{C^3}{C^2 - \lambda^2} B = \frac{1}{\pi}\left(\frac{\alpha_3}{\alpha_1\alpha_2}\right)^2 \frac{\cos\pi\lambda(1+2\beta) - \cos\pi\lambda}{\lambda \sin\pi\lambda} \,. \tag{A.15}$$

Substituting the special value $\lambda = i\mu\alpha_3$, the last identity can be recast as

$$B^T \frac{C^3}{C_3^2} B = -\frac{2}{\pi\mu\alpha_3}\left(\frac{\alpha_3}{\alpha_1\alpha_2}\right)^2 (\coth\pi\mu\alpha_1 + \coth\pi\mu\alpha_2)^{-1} \,. \tag{A.16}$$

Note that the large μ asymptotic behavior is very sensitive to the direction in which infinity is approached. In particular, if it is approached in the positive direction, which is the usual case of interest, the last expression reduces to $-\frac{1}{\pi\mu\alpha_3}\left(\frac{\alpha_3}{\alpha_1\alpha_2}\right)^2$ with exponential precision.

Some additional useful identities are

$$\sum_{r=1}^{3} \frac{1}{\alpha_r} A^{(r)} C A^{(r)T} = 0\,, \tag{A.17}$$

$$\sum_{r=1}^{3} \alpha_r A^{(r)} C^{-1} A^{(r)T} = \frac{\alpha}{2} B B^T\,, \tag{A.18}$$

where we have introduced

$$\alpha = \alpha_1 \alpha_2 \alpha_3\,. \tag{A.19}$$

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